

Combining Breadth-First and Depth-First Strategies in Searching for Treewidth

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Abstract

Breadth-first and depth-first search are basic search strategies upon which many other search algorithms are built. In this paper, we describe an approach to integrating these two strategies in a single algorithm that combines the complementary strengths of both. We report preliminary computational results using the treewidth problem as an example.

Introduction

Breadth-first and depth-first search are basic search strategies upon which many other search algorithms are built. Given the very different way in which they order node expansions, it is not obvious that they can be combined in the same search algorithm. In this paper, we describe an approach to integrating these two strategies in a single algorithm that combines the complementary strengths of both. To illustrate the benefits of this approach, we use the treewidth problem as an example.

The treewidth of a graph (also known as the induced treewidth) is a measure of how similar the graph is to a tree, which has a treewidth of 1. A completely connected graph is least similar to a tree, and has a treewidth of $n - 1$, where n is the number of vertices in the graph. Most graphs have a treewidth that is somewhere in between 1 and the number of vertices minus 1.

There is a close relationship between treewidth and vertex elimination orders. Eliminating a vertex of a graph is defined as follows: an edge is added to every pair of neighbors of the vertex that are not adjacent, and all the edges incident to the vertex are removed along with the vertex itself. A vertex elimination order specifies an order in which to eliminate all the vertices of a graph, one after another. For each elimination order, the maximum degree (i.e., the number of neighbors) of any vertex when it is eliminated from the graph is defined as the width of the elimination order. The treewidth of a graph is defined as the minimum width over all possible elimination orders, and an optimal elimination order is any order whose width is the same as the treewidth.

Many algorithms for exact inference in Bayesian networks are guided by a vertex elimination order, including Bucket Elimination (Dechter 1999), Junction-tree elimination (Lauritzen & Spiegelhalter 1988), and Recursive Conditioning (Darwiche 2001). In fact, the complexity of all of

these algorithms is exponential in the treewidth of the graph induced by the network. For these algorithms, use of a sub-optimal elimination order leads to inefficiency, and improving an elimination order by even small amount can result in large computational savings. Solving the treewidth problem exactly, and finding an optimal elimination order, allows these algorithms to run as efficiently as possible.

Previous work

Finding the exact treewidth of a general graph is an NP-complete problem (Arnborg, Corneil, & Proskurowski 1987). One approach to finding the exact treewidth is depth-first branch-and-bound search in the space of vertex elimination orders (Gogate & Dechter 2004). However, Dow and Korf (2007) recently showed that best-first search can dramatically outperform depth-first branch-and-bound search by avoiding repeated generation of duplicate search nodes.

In the search space of the treewidth problem, each node corresponds to an *intermediate graph* that results from eliminating a set of vertices from the original graph. Figure 1 shows the treewidth search space for a graph of 4 vertices. Each oval represents a search node that is identified by the set of vertices eliminated so far from the original graph. A path from the start node (which has an empty set of eliminated vertices) to the goal node (which has all vertices eliminated) corresponds to an elimination order, and there is a one-to-one mapping from the set of elimination orders to the set of paths from the start to the goal node. Although there are $n!$ different elimination orders for a graph of n vertices, there are only 2^n distinct search nodes. This is because different ways of eliminating the same set of vertices always arrive at the same intermediate graph (Bodlaender *et al.* 2006), and there is only one distinct intermediate graph for each combination (as opposed to permutation) of the vertices. Depth-first branch-and-bound search treats the search space as a tree with $n!$ distinct states instead of a graph with only 2^n states. The faster performance of best-first treewidth search reflects the difference in size between a search tree and a search graph (Dow & Korf 2007).

Unfortunately, the scalability of best-first (treewidth) search is limited by its memory requirements, which tend to grow exponentially with the search depth. To improve scalability, Dow and Korf use a memory-efficient version of best-first search called *breadth-first heuristic search* (Zhou

& Hansen 2006), which, like *frontier search* (Korf *et al.* 2005), only stores the search frontier in memory and uses a divide-and-conquer approach to reconstruct the solution path after the goal is reached. In fact, they use a variant of breadth-first heuristic search, called *SweepA** (Zhou & Hansen 2003), that exploits the fact that the search graph for the treewidth problem is a partially ordered graph. A *partially ordered graph* is a directed graph with a layered structure, such that a node in one layer can only have successors in the same layer or later layers. This allows layers to be removed from memory after all their nodes are expanded. *SweepA** expands all the nodes in one layer before considering any nodes in the next layer, and uses an admissible heuristic and an upper bound to prune the search space.

Besides exploiting the layered structure of the search graph using *SweepA**, there is another important way in which the search algorithm of Dow and Korf limits use of memory. Because the size of an intermediate graph can vary from several hundred bytes to a few megabytes, storing an intermediate graph at each search node is impractical for all but the smallest problems. Instead, Dow and Korf store with each search node only the set of vertices that have been eliminated so far. Each time a node is expanded, its corresponding intermediate graph is generated on-the-fly by eliminating from the original graph those vertices stored with the node. While this approach is space-efficient, it incurs the overhead of intermediate graph generation every time a node is expanded. For large graphs, this overhead is considerable. In the rest of this paper, we describe a technique that eliminates much of this overhead.

Meta search space

To reduce the time overhead of intermediate graph generation, we describe a search algorithm that does not generate the intermediate graph from the original graph at the root node of the search space. Instead, it generates it from the intermediate graph of a close neighbor of the node that is being expanded. The advantage is that the intermediate graph of a close neighbor is already very similar, and so there is much less overhead in transforming it into a new intermediate graph. The simplest way to find the node’s closest neighbor is by computing shortest paths from a node to all of its neighbors and picking the closest one. But at first, this does not seem to work for the treewidth problem, since its state space is a partially ordered graph in which the distance between any pair of nodes at the same depth is infinite. For example, suppose there are two nodes that correspond to the intermediate graphs that result from eliminating vertices $\{1, 2, 3\}$ and $\{1, 2, 4\}$ from the original graph, respectively. From Figure 1, one can see that there is no legal path between these two nodes, because once a vertex is eliminated, it cannot be “uneliminated.”

Our idea is to measure the distance between a pair of nodes in a *meta search space*, instead of the original search space. A meta search space has exactly the same set of states as the original search space, but is augmented with a set of meta actions that can transform one node into another in ways not allowed in the original search space. For example, a meta action for the treewidth problem can be an action

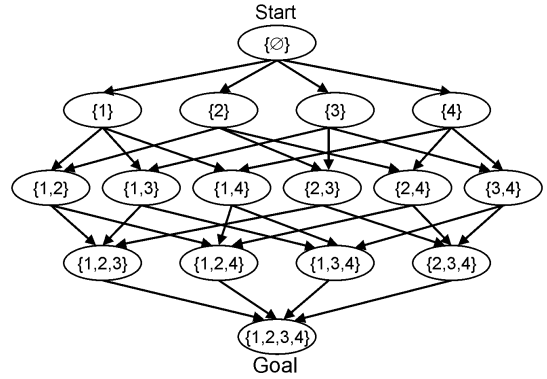


Figure 1: The search space of treewidth for a graph of 4 vertices. Each oval represents a search node identified by the set of vertices eliminated so far. The start node corresponds to the original graph with an empty set of eliminated vertices and the goal node is the one with all the vertices eliminated.

that “uneliminates” a vertex by reversing the changes made to a graph when the vertex was eliminated. For the treewidth problem augmented with the “uneliminate” meta action, its search graph is an undirected version of the graph shown in Figure 1. In this new graph, called a meta search graph, actions (i.e., edges) can go back and forth between a pair of adjacent nodes, and this allows us to generate the intermediate graph of a node from another node at the same depth. As we will see, this is very useful for breadth-first heuristic search (Zhou & Hansen 2006), which expands nodes in order of their depth in the search space.

Since a node is uniquely identified by the set of vertices eliminated, we use the same lower-case letter (e.g., $n, u,$ and v) to denote both a node and a set of eliminated vertices in the rest of this paper. To implement the “uneliminate” meta action, each edge of the meta search graph is labeled by a tuple $\langle u, v, \Delta E^+, \Delta E^- \rangle$, where u (v) is the set of vertices eliminated so far at the source (destination) node of the edge, and ΔE^+ (ΔE^-) is the set of edges added to (deleted from) the graph when the vertex in the singleton set $v \setminus u$ is eliminated. Let $G_n = \langle V_n, E_n \rangle$ be the intermediate graph associated with node n . The task of adding a previously-eliminated vertex back to the graph can be expressed formally as: given $G_v = \langle V_v, E_v \rangle$ and $e = \langle u, v, \Delta E^+, \Delta E^- \rangle$, how to compute $G_u = \langle V_u, E_u \rangle$? Since all the changes are recorded with the edge e , one can reconstruct $G_u = \langle V_u, E_u \rangle$ as follows,

$$V_u = V_v \cup v \setminus u \quad (1)$$

$$E_u = E_v \cup \Delta E^- \setminus \Delta E^+ \quad (2)$$

That is, by adding (deleting) the edges that have been previously deleted (added) to the graph, the “uneliminate” meta action can undo the effects of an elimination action in the original search space.

In general, adding meta actions can turn directed search graphs into undirected graphs such that the effects of any action in the original search space can be “undone.” This guarantees that any changes made to the current state (e.g., the intermediate graph) is reversible, creating a graph with the

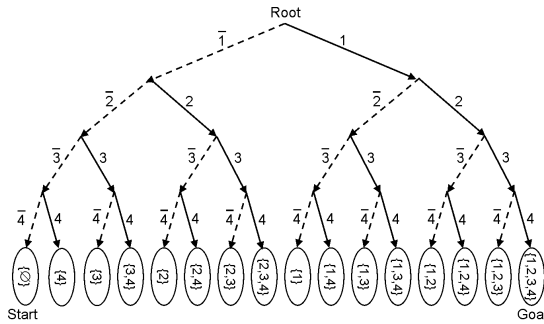


Figure 2: Example binary decision tree with search nodes for treewidth as leaves. For simplicity, non-leaf nodes are not shown.

following appealing property: for *any* two states reachable from the start state, there is *always* a path that maps one into the other. This property allows a search algorithm to generate the state representation of a node from *any stored node*, because if actions all have deterministic effects, then a state s' is uniquely identified by another state s plus a path from s to s' . If it takes less space to represent a path between s and s' , then this approach to state encoding can save memory, although at the cost of some computational overhead.

For the treewidth problem, this means the intermediate graph of a node can be generated from any node instead of only from the node's direct ancestors such as the start node. Thus, one only needs to maintain a single intermediate graph, which can be modified to become the intermediate graph for any node in the search space. An interesting question is how to minimize the overhead of generating the intermediate graph from one node to another. The answer depends on the search strategy, because ultimately our goal is to minimize the overhead of expanding not just a single node but a set of nodes.

Decision-tree representation of frontier nodes

The solution we propose is to use an ordered decision tree to store the set of frontier nodes at the current depth of breadth-first heuristic search as the leaves of the decision tree. Unlike explicit-state search methods, this approach can be viewed as a variant of symbolic search in which the state-representation similarities among a set of nodes are retained in the decision tree and exploited by the search algorithm.

A decision tree is defined as a rooted tree in which every non-leaf node is a decision node that performs a test on a variable, the value of which is then used to determine recursively the next decision node until a leaf node is reached. A decision tree is commonly used to represent a discrete function over a set of variables. For the treewidth problem, these are Boolean variables, one for each vertex. A Boolean variable has the value of true if its corresponding vertex has been eliminated. Thus, we only need to focus on binary decision trees here, even though multi-valued decision trees would be needed for the general case.

To make operations on decision trees more efficient, an ordering constraint is usually enforced that requires the order in which variables are tested be the same on any path

from the root to a leaf node. The resulting data structure is called an ordered binary decision tree, and an example is shown in Figure 2. In this example, variables are tested in increasing order of vertex number: 1, 2, 3, then 4. A solid (dashed) edge represents a truth (false) assignment to the variable being tested at the source node of the edge. A leaf node corresponds to a complete assignment to all the variables, and there is a one-to-one mapping from the set of leaf nodes shown in Figure 2 to the set of search nodes shown in Figure 1. From now on, we call a decision trees that stores the frontier nodes of a search graph in this way a *frontier decision tree*.

To support meta actions, each edge of the frontier decision tree stores “undo” information as needed by Equations (1) and (2). Note that for an undirected search space, there is no need to store such information.

Space overhead We use a frontier decision tree in order to reduce the time overhead for regenerating the intermediate graph from the root node. But this must be weighed against the space overhead for storing the decision tree. We make a few comments here about how to minimize the space overhead, and discuss this important question further near the end of the paper.

An important way to save space in a frontier decision tree is to find a good ordering of the variables, which affects the number of decision nodes needed to represent a set of frontier nodes. While finding an optimal ordering is a hard combinatorial optimization problem in itself, good orders can often be found quickly by using simple heuristics. We tried three variable-ordering heuristics, the details of which are described in the computational results section.

The amount of memory needed to store a frontier decision tree depends not only on how many frontier nodes are stored, but also on the number of non-leaf decision nodes. To improve memory efficiency, we would like to remove a decision node if it does not lead to any frontier node. To allow pruning of useless decision nodes, we store a leaf-node counter at each decision node in the tree. Each time a leaf node is deleted, all of its ancestor decision nodes decrease their leaf-node counters by one, and a decision node is deleted as soon as its leaf-node counter reaches zero. With this pruning rule in place, it is easy to prove that the asymptotic space complexity of a frontier decision tree is $O(VN)$, where V is the number of vertices in the original graph and N is the number of frontier nodes stored in the decision tree. This result holds even in the worst case in which no two leaf nodes share the same sub-path in the frontier decision tree. Given that the space complexity of an explicit-state (as opposed to symbolic) representation is also $O(VN)$, it follows that our decision-tree representation of frontier nodes does not increase the asymptotic space complexity of the search algorithm. But because it stores additional information such as common prefixes in the tree, it can increase the space complexity of the search by a constant factor. We discuss this potential space overhead and techniques for reducing it later in the paper. For now, we focus on using this frontier decision tree to improve the time complexity of search.

Depth-first search in frontier decision tree

The purpose of using an ordered binary decision tree to store the set of frontier nodes is twofold. First, it reveals the similarities among the frontier nodes, because nodes with the same prefix (according to the test ordering) share the same ancestor node in the decision tree. For example, because nodes $\{1, 2, 3\}$ and $\{1, 2, 3, 4\}$ share the same prefix $\{1, 2, 3\}$, they have the same parent node in the decision tree. On the other hand, because nodes $\{\emptyset\}$ and $\{1, 2, 3, 4\}$ have nothing in common, their common ancestor is only the root node. Second, the tree topology guarantees there is a unique path from the root to a leaf node. This facilitates the use of a tree-search algorithm such as depth-first search to determine the order in which frontier nodes are expanded.

It is well-known that depth-first search has excellent memory-reference locality. This is particularly well suited for decision trees, since a depth-first search of a decision tree always visits nodes with the same prefix before visiting nodes with different prefixes, and the *longer* the prefix that is shared by two nodes, the *closer* they will be visited in depth-first search. For the treewidth problem, this means that if two nodes have similar intermediate graphs, they will be expanded close to each other, and the more similar their intermediate graphs, the closer together they will be expanded. To minimize the intermediate-graph generation overhead for the entire set of frontier nodes at the current search depth, we use depth-first traversal of the decision tree, which visits all leaf nodes of the decision tree. Thus, our treewidth algorithm adopts a hybrid search strategy that uses depth-first traversal in a symbolic (e.g., decision-tree) representation of the (meta) search graph to determine the order of node expansions for the current depth of breadth-first heuristic search (or nodes with the same minimum f -cost in the case of A*). The depth-first search aspect essentially serves as a tie-breaking strategy in breadth-first (or best-first) search to improve its memory-reference locality; in the case of treewidth computation, it also reduces the overhead of generating the intermediate graphs.

Example Figure 3 shows an example how depth-first search can reduce the intermediate-graph generation overhead in breadth-first heuristic search for treewidth. Suppose the three leaves shown in the figure are the frontier nodes of breadth-first heuristic search, and the intermediate graph has already been generated for node $\{1, 3, 4\}$. Depth-first search will visit node $\{1, 2, 4\}$ next, and then node $\{1, 2, 3\}$. A sequence of dark (solid and dashed) arrows represents the order in which actions are taken to “migrate” the intermediate graph from node $\{1, 3, 4\}$ to node $\{1, 2, 4\}$ and then to node $\{1, 2, 3\}$. A solid dark arrow moving towards the leaf (root) represents an action that eliminates (uneliminates) a vertex. Dashed arrows represent no-op actions that simply move an intermediate graph around in the decision tree without changing its content. The action sequence shown in Figure 3 starts with a meta action (shown as a dark, upward arrow from node $\{1, 3, 4\}$) that “uneliminates” vertex 4 from the intermediate graph $G_{\{1,3,4\}}$ in order to generate $G_{\{1,3\}}$. Then vertex 3 is “uneliminated” to generate $G_{\{1\}}$. Next, ver-

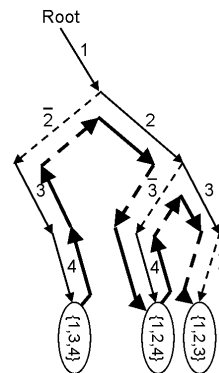


Figure 3: Depth-first search in a binary decision tree can be used to order node expansions in breadth-first treewidth computation.

tex 2 is eliminated from the intermediate graph to generate $G_{\{1,2\}}$, and then vertex 4 is eliminated to arrive at $G_{\{1,2,4\}}$. To migrate from node $\{1, 2, 4\}$ to node $\{1, 2, 3\}$, vertex 4 is “uneliminated” and then vertex 3 is eliminated to generate the intermediate graph $G_{\{1,2,3\}}$. Because an “uneliminate” meta action does not need to check for connectivity between all possible pairs of a vertex’s neighbors, it is usually much cheaper than eliminating a vertex. Thus, we only count the number of times an elimination action is performed as overhead. In this example, there are altogether 3 elimination actions. For comparison, generating the intermediate graphs for nodes $\{1, 2, 3\}$ and $\{1, 2, 4\}$ from the root node would require 6 elimination actions (3 for node $\{1, 2, 3\}$ and 3 for node $\{1, 2, 4\}$), which is (almost) twice as expensive.

Note that the benefit of our approach increases as the search frontier moves further away from the root node. For example, if the three leaf nodes in Figure 3 are a hundred elimination steps away from the root node, then it will take about 200 elimination actions to regenerate the intermediate graphs for nodes $\{1, 2, 4\}$ and $\{1, 2, 3\}$; whereas it still takes the same number of elimination actions (3) with our approach, no matter how deep these nodes are, as long as the intermediate graph has just been generated for a node ($\{1, 3, 4\}$ in this example) that is close by. Moreover, the overhead of elimination actions can differ significantly depending on the size and topology of the intermediate graph. Because intermediate graphs have fewer vertices as the search goes deeper, elimination actions tend to become cheaper as frontier nodes move further away from the root. In other words, the overhead of the 3 elimination actions needed in our approach is probably cheaper than 1.5% (i.e., $3/200$) of the overhead incurred by generating the intermediate graphs from the root, if the overhead differences in elimination actions are accounted for.

Computational results

Both random graphs and benchmark graphs were used in our experiments. Given the number of vertices V , a random graph is generated by selecting a fixed number of edges E uniformly from the set of $V(V - 1)/2$ possible edges. In our experiments, we used $V = 35$ and $E = 140$ to generate

Tw	DFBnB		BFHS	
	Exp	Sec	Exp	Sec
15	15,100,721	2,815.7	535,991	37.2
14	17,775,261	2,970.8	340,206	24.3
14	7,470,073	1,160.4	175,678	17.0
15	4,650,125	882.7	341,395	20.5
15	7,057,634	1,302.7	330,548	17.6

Table 1: Comparison of the QuickBB treewidth solver based on depth-first branch-and-bound (DFBnB), and a treewidth solver based on breadth-first heuristic search (BFHS). Columns show treewidth (Tw), number of node expansions (Exp) and running time in CPU seconds (Sec).

100 random graphs, which correspond to the most difficult set of random graphs used in (Dow & Korf 2007). All experiments were run on a machine with two Intel 2.66 GHz Xeon dual-core processors with 8 GB of RAM, although the search algorithm never used more than 1 GB of RAM and no multi-threading parallelization was used.

Recall the order in which decision variables are tested can affect the size of an ordered decision tree. We tested three variable-ordering heuristics: (a) the random ordering heuristic, (b) the minimum-degree-vertex-first heuristic, which orders the variables in increasing degrees of their corresponding vertices, and (c) the maximum-degree-vertex-first heuristic, which does the opposite. Our experiments indicate that the random ordering and minimum-degree-vertex-first heuristics store on average 40% and 135% more decision nodes than the maximum-degree-vertex-first heuristic in solving the set of 100 random graphs, respectively. Thus, the maximum-degree-vertex-first heuristic was used to produce all the experimental results reported next.

Next we studied two different strategies for caching the “undo” information at the edges of the frontier decision tree. The *cache-until-removal* strategy stores “undo” information for every edge of the decision tree until the edge is removed due to the pruning of some decision node. The *cache-until-backtrack* strategy stores “undo” information until the depth-first traversal of the decision tree backtracks from the edge to the source decision node of that edge. In other words, it only stores “undo” information along the current “stack” of the depth-first traversal. Thus, the maximum number of edges for which “undo” information is stored cannot exceed the depth of the decision tree, which is bounded by the number of vertices in the original graph. Because the memory requirements depend on the complexity of the “undo” information measured in terms of the size of ΔE^+ and ΔE^- , our implementation keeps track of the maximum number of edges included in all such ΔE sets, which reflects accurately the total amount of memory used for storing “undo” information over the entire decision tree. With the *cache-until-removal* strategy, the average peak number of ΔE edges cached is 7,253,520 edges. This number decreased to about 405 edges when the *cache-until-backtrack* strategy was used, reducing the number of ΔE edges by a factor of over 17,900 times! Surprisingly, this has little effect on the average running time of the algorithm; using the *cache-until-backtrack* strategy increased the average running time by less than

Tw	Start node		Neighbor		Sec ratio
	Exp	Sec	Exp	Sec	
15	535,991	37.2	517,157	25.0	149%
14	340,206	24.3	322,498	15.7	155%
14	175,678	17.0	160,038	9.0	189%
15	341,395	20.5	334,150	14.2	144%
15	330,548	17.6	327,360	13.3	132%
16	4,776,876	433.8	4,677,177	230.8	188%
16	3,944,331	431.1	3,826,379	211.8	204%
16	3,767,825	396.4	3,629,411	193.9	204%
17	6,324,117	682.3	6,232,333	342.5	199%
16	4,520,092	404.0	4,431,308	215.8	187%

Table 2: Comparison of treewidth solvers based on breadth-first heuristic search with different approaches to generating the intermediate graphs. The column labeled “Start node” corresponds to the approach of generating the intermediate graph by eliminating vertices from the original graph; the column labeled “Neighbor” corresponds to the approach of modifying the intermediate graph of a neighboring node. The horizontal line separates the 5 easiest instances from the 5 hardest ones in a set of 100 random graphs.

1.7%, which is hardly noticeable. We also compared the treewidth solution and the number of node expansions of the two caching strategies for all 100 instances to make sure the results are correct. Results in the rest of this section were obtained by using the *cache-until-backtrack* strategy only.

We used breadth-first heuristic search as the underlying search algorithm, which needs an upper bound to prune nodes with an f -cost greater than or equal to the upper bound. While an upper bound for treewidth can be quickly computed by using the minimum fill-in (min-fill) heuristic, in our experiments we used divide-and-conquer beam search (Zhou & Hansen 2004a) that can usually find tighter upper bounds. A beam-search variant of breadth-first heuristic search, divide-and-conquer beam search (DCBS) limits the maximum size of a layer in the breadth-first search graph. When memory is full (or reaches a predetermined bound), DCBS recovers memory by pruning the least-promising nodes (i.e., the nodes with the highest f -cost) from the Open list before it continues the search.

Using a memory bound of 64 nodes, DCBS finds the exact treewidth for 97 out of the 100 random graphs. For the remaining three graphs, its solutions are very close to the exact treewidth. When averaged over all 100 graphs, the solution found by DCBS is within 0.2% of optimality. Since our current implementation does not perform divide-and-conquer solution reconstruction, a process that can potentially improve the solution quality by solving multiple subproblems of the original treewidth problem, the solution quality found by a full-fledged implementation of DCBS can be better. However, even with our current implementation, we were able to find better solutions than QuickBB (Gogate & Dechter 2004) on several benchmark graphs, including a network in the Bayesian Network Repository called *pigs* with 441 vertices and 806 edges for which DCBS found a treewidth of 9, improving on the best solution found so far for this problem.

Graph	Ub	Lb	Tw	Stored	Exp	Sec
queen5_5	18	12	18	961	1,294	0.1
david	13	10	13	483	2,009	0.4
queen6_6	25	15	25	11,995	13,353	1.6
miles500	22	21	22	2	2	2.3
inithx.i.1	56	55	56	209	370	30.8
queen7_7	35	18	35	597,237	935,392	149.6
myciel5	19	14	19	678,540	3,418,309	192.3

Table 3: Performance of breadth-first heuristic search on benchmark graphs. Depth-first traversal in a frontier decision tree is used to reduce the overhead in generating the intermediate graphs. Columns show the upper bound found by divide-and-conquer beam search (Ub), the heuristic value for the start node (Lb), the treewidth (Tw), the peak number of frontier nodes stored (Stored), the number of node expansions (Exp), and running time in CPU seconds (Sec).

Next we compared QuickBB, a treewidth solver based on depth-first branch-and-bound (DFBnB) search, and our implementation of breadth-first heuristic treewidth (Dow & Korf 2007), a solver based on breadth-first heuristic search (BFHS). The 5 instances shown in Table 1 are the ones both DFBnB and BFHS can solve among the same 100 random graphs we tested before. These five instances are among the easiest in this test set, and DFBnB cannot solve the other 95 instances. BFHS solved all 100 instances and spent about 154 seconds per instance (for comparison, the average time is only 23.3 seconds for BFHS in Table 1). Even for the 5 easiest instances, BFHS ran about 76.7 times faster than DFBnB on average. These are basically the results obtained by (Dow & Korf 2007) (possibly enhanced by the tighter upper bounds found by DCBS), and we will use them as a baseline to show the advantage of our hybrid algorithm.

The BFHS results shown in Table 1 are based on the naive way of generating the intermediate graph from the original graph, which can be further improved. Table 2 shows the comparison of two different approaches to generating the intermediate graphs, one by eliminating vertices from the original graph and the other by modifying the intermediate graph of a neighboring node. Both approaches use BFHS as their underlying search algorithm. Note that the ratio by which the second approach improves on the first one increases with the hardness of the instance. For example, the average speedup ratio for the 5 easiest instances (shown above the horizontal line) is 1.5; for the 5 hardest instances (shown below the horizontal line), it is 2.0. For reference, the average speedup ratio is about 1.75 for all 100 instances. The reason our hybrid algorithm performs increasingly better on hard instances is that the more nodes are expanded, the easier it is to find a neighbor whose intermediate graph closely resembles the one to be generated next.

It is worth pointing out that the admissible heuristic for treewidth, which computes the maximum degree of a min-degree vertex encountered in a series of edge contractions (see (Gogate & Dechter 2004) for details), is expensive to compute. Thus, the speedup ratios (measured as CPU-second ratios) shown in Table 2 are diluted by the time it takes to compute the heuristic, which is the same no matter

Graph	QBB	QBB ⁰⁷	BFHT ⁰⁷
queen5_5	1.5	2.8	1.8
david	3.2	5.1	3.5
queen6_6	46	101.8	8
miles500	fault	149.7	mem
inithx.i.1	42.7	39.4	514.2
queen7_7	> 5 hrs	> 5 hrs	1872.9
myciel5	fault	6699.6	1071.3

Table 4: Performance comparison of existing treewidth solvers on benchmark graphs. Columns show the running time in CPU seconds for (i) QuickBB on our machine (QBB), (ii) QuickBB as previously reported by Dow & Korf (2007) (QBB⁰⁷), and (iii) breadth-first heuristic treewidth as previously published by Dow & Korf (2007) (BFHT⁰⁷). Results under columns labeled QBB⁰⁷ and BFHT⁰⁷ were obtained on the same machine. The string “mem” indicates the corresponding solver required more than 800 megabytes of memory, and “fault” indicates the solver terminated prematurely before finding an optimal solution on our machine.

how intermediate graphs are generated. When excluding the time it takes to compute the heuristic, the average speedup ratio increases from 1.75 to 2 on the random graphs.

Table 3 shows the performance of our hybrid algorithm on benchmark graphs for DIMACS graph coloring instances. Compared to published results (Dow & Korf 2007; Gogate & Dechter 2004) shown in Table 4, Table 3 shows improvement over the state of the art. The running time shown in Table 3 includes the CPU seconds for computing the upper bound using DCBS.

We haven’t mentioned yet the additional space overhead of the decision-tree representation of frontier nodes. In our experiments, it increased the memory requirements of the treewidth search algorithm by a factor of up to ten times compared to the strategy of regenerating the intermediate graph from the root node. Although this is still much better than storing a copy of the intermediate graph in each node, it is a serious drawback compared to the approach of always regenerating the intermediate graph from the root node, especially since memory is a limiting factor for graph search. The reduction in search time we achieved may not be worth this much of an increase in memory overhead. However we believe there are many opportunities for substantially reducing the space overhead of the frontier decision tree, and we plan to explore these in future work.

Conclusion and future work

We have presented a novel combination of breadth-first and depth-first search that allows a single search algorithm to possess the complementary strengths of both. While our paper focuses on the treewidth problem, many of the ideas have the potential to be applied to other search problems, especially graph-search problems with large encoding sizes, for which memory-reference locality is the key to achieving good performance. Possibilities include model checking (Clarke, Grumberg, & Peled 2000), where a large data structure that represents the current state is typically stored

with each search node, and constraint-based planning and scheduling (Ruml, Do, & Fromherz 2005), where a simple temporal network is stored with each search node. As long as the similarities among different search nodes can be captured in a form that allows depth-first search to exploit the state-representation locality in node expansions, the approach we have described could be effective.

However, it is very important to find ways to further reduce the space overhead of the frontier decision tree. A promising approach is to use structured duplicate detection (Zhou & Hansen 2004b). The idea is for the decision tree to only store the small fraction of the frontier nodes that have the same *duplicate-detection scope* as the node being expanded; the rest of the frontier nodes can be stored much more compactly using a bit-vector representation. In this approach, the frontier decision tree would be built incrementally and only used to encode those frontier nodes that are on or near the decision-tree path currently being explored by the depth-first search. Just as the peak number of ΔE edges stored can be reduced by over four orders of magnitude by only storing those edges that are on the stack of the depth-first search, it seems promising to adopt this similar approach to reducing the peak number of internal decision nodes stored in the frontier decision tree.

Finally, our experiments show that the memory overhead of the frontier decision tree tends to decrease with problem size, while the time savings increases. Thus the potential benefit of our approach increases with problem size. Moreover, if we use an external-memory search algorithm to solve very large treewidth problems, it is likely to be more important to save time than to save memory. More work is needed to fully evaluate the potential of this approach and the space/time tradeoff it offers.

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