Joint Optimization of Communication Rates and Linear Systems

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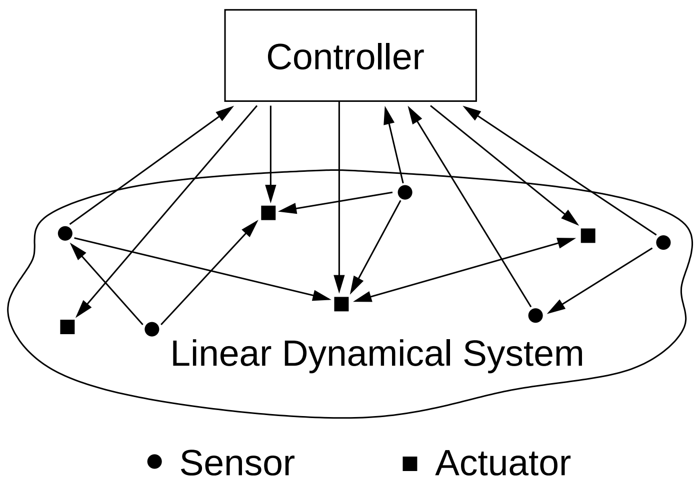
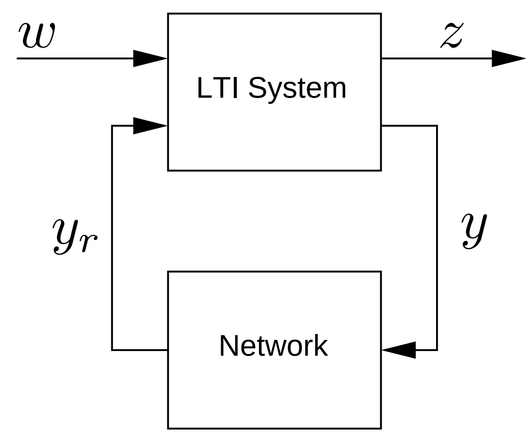
Motivation: Networked control and estimation

Linear system closed over communication network





- w is exogenous input; z is critical output
- examples: sensor networks, distributed control

Problem statement

Goal: optimize stationary performance of linear system

Design variables:

- network parameters: powers, bandwidths, routing, . . .
- linear system parameters: gains, . . .

We model

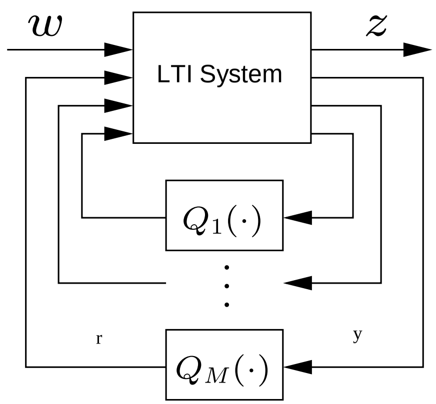
- quantization errors due to bit-rate limitations
- bit-rate limitations due to channel capacities and routing

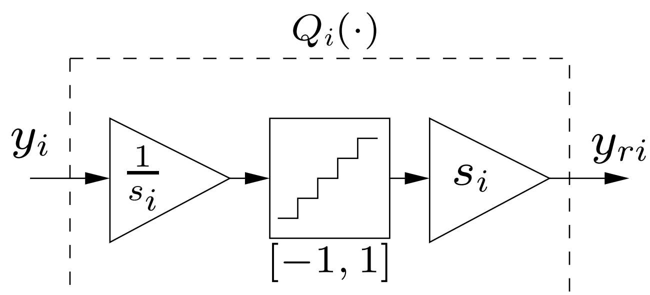
but ignore

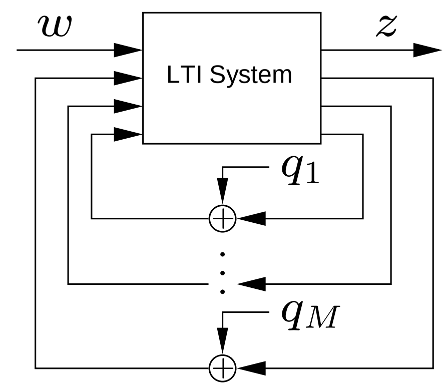
- delay variations, packet loss, transmission errors, asynchronicity

Quantization error from bit-rate limitations

We focus on bit-rate constraints imposed by communication channels:



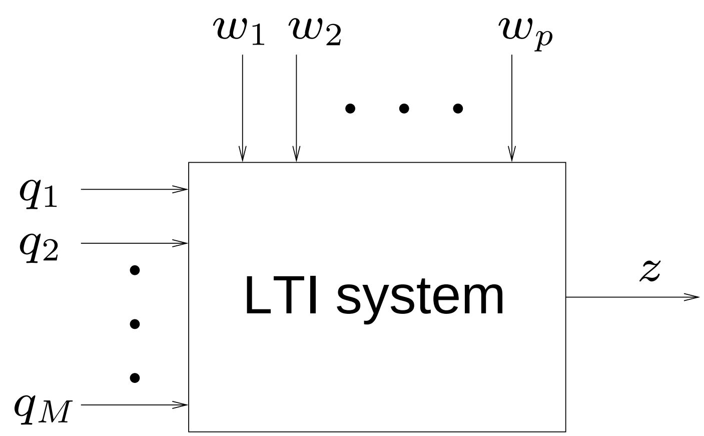




- uniform quantization on the range $[-1, 1]$: b_i bits/channel
- scaling factor s_i depends on range of signal y_i, *e.g.*, $s_i = 3\,\mathbf{rms}(y_i)$
- white noise model of quantization errors:

$$\mathbf{E}\ q_i^2 = \frac{1}{3} s_i^2 2^{-2b_i}$$

How bit-rates affect linear system performance



Performance (output variance) depends on bit-rate allocation

$$\lim_{t\to\infty} \mathbf{E}\,\|z\|_2^2 = \sum_{i=1}^{M} \|G_i\|_2^2 \frac{1}{3} s_i^2 2^{-2b_i} + \sigma_{zw}^2$$

- G_i: transfer function from q_i to z
- σ_{zw}^2: variance induced by exogenous input w

Bit-rate constraints

quantization bit allocation b_i and communication rate r_i proportional

General convex model of bit-rate constraints

$$
\begin{array}{rcll}
f_k(b,\theta) & \leq & 0 & \textit{(capacity constraints)} \\
h_k^T\theta & \leq & d_k & \textit{(resource limits)} \\
\theta_k & \geq & 0 & \textit{(nonnegative constraints)} \\
\underline{b}_k \leq b_k & \leq & \overline{b}_k & \textit{(lower and upper bounds)}
\end{array}
$$

- θ network variables: powers, bandwidths, time fractions, routing, . . .
- f_k convex in (b,θ), monotone increasing in b, monotone decreasing in θ
- h_k have nonnegative entries; d_k positive, resource limits
- $\underline{b}_k$ to ensure white noise model reasonable; $\overline{b}_k$ hardware limitations

ignore (for now): b_k are integers

Examples of bit-rate constraints

- Parallel Gaussian channels (MAC or BC with FDMA)

$$b_i \leq cW_i \log_2\left(1 + \frac{P_i}{N_i W_i}\right) \qquad i = 1, \ldots, M$$

- Gaussian Multiple access channel with CDMA

$$\sum_{i \in Z} b_i \leq cW \log_2\left(1 + \frac{\sum_{i \in Z} P_i}{NW}\right) \quad \text{for all} \quad Z \subseteq \{1, 2, \ldots, M\}.$$

- Broadcast channel with TDMA

$$b_i \leq \tau_i cW \log_2\left(1 + \frac{P}{N_i W}\right), \qquad \sum_{i=1}^{M} \tau_i = 1$$

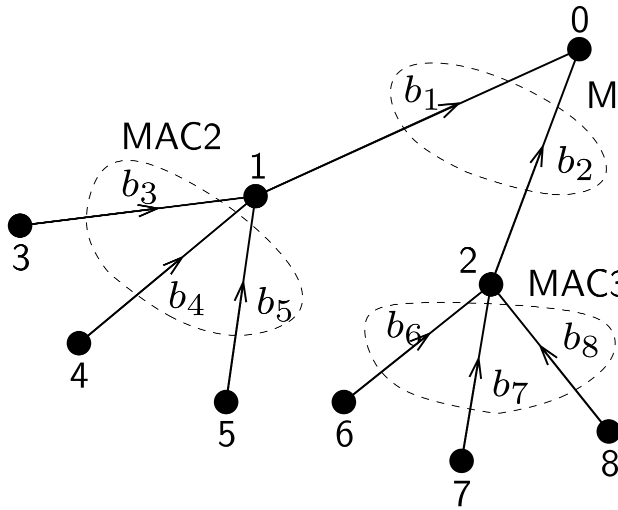
- total power, bandwidth constraints

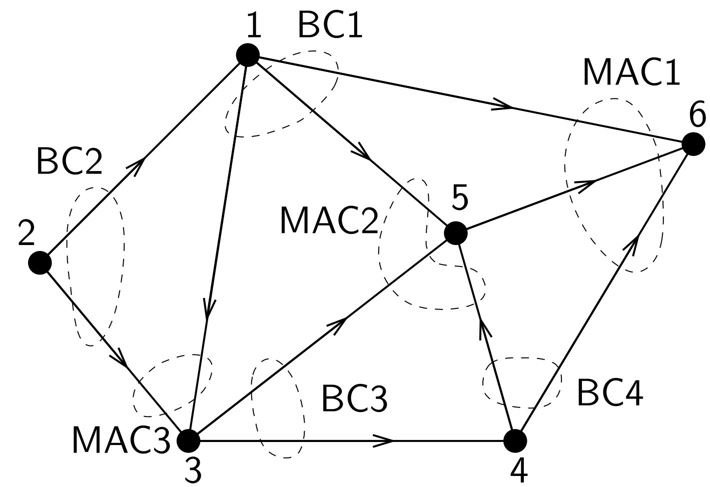
$$P_1 + \cdots + P_M \leq P_{\mathrm{tot}}, \qquad W_1 + \cdots + W_M \leq W_{\mathrm{tot}}.$$

jointly convex in bit-rates b_i, powers P_i, and bandwidths W_i

Networks

Constraints imposed by network topology also **convex**





Example: Forwarding network with tree topology (left)

$$b_3 + b_4 + b_5 \leq b_1, \qquad b_6 + b_7 + b_8 \leq b_2$$

Can have individual constraints, or share resources within groups:

$$P_i \leq P, \qquad W_3 + W_4 + W_5 \leq W$$

Resource allocation via convex optimization

allocate resources (powers, bandwidths, etc.) to minimize quantization induced variance, subject to capacity constraints, resource limitations, network topology, routing

$$\begin{array}{lll} \text{minimize} & \sum_{i=1}^{M} a_i 2^{-2b_i} & \textit{(quantization induced variances)} \\ \text{subject to} & f_k(b,\theta) \leq 0 & \textit{(capacity constraints)} \\ & h_k^T \theta \leq d_k & \textit{(resource limits)} \\ & \theta_k \geq 0 & \textit{(nonnegative constraints)} \\ & \underline{b}_k \leq b_k \leq \overline{b}_k & \textit{(lower and upper bounds)} \end{array}$$

we ignore (for now) $b_i \in \mathbf{Z}$

convex problem, hence readily solved

can exploit special structure: separable except for a few coupling constraints

Dual decomposition method

convex (primal) problem, separable except for few coupling constraints:

$$\begin{array}{ll} \text{minimize} & \sum_i f_i(x_i) \\ \text{subject to} & x_i \in X_i, \quad \sum_i A_i x_i \preceq b \end{array}$$

dual problem:

$$\begin{array}{ll} \text{maximize} & g(\lambda) \\ \text{subject to} & \lambda \succeq 0 \end{array}$$

where

$$g(\lambda) = \sum_{i=1}^{M} \left\{ \min_{x_i \in X_i} \left(f_i(x_i) + \lambda^T A_i x_i \right) \right\} - \lambda^T b$$

- each λ_i associated with a coupling constraint
- can recover primal solution by solving dual

Solving dual problem

- can use cutting-plane or other method
- can evaluate $g(\lambda)$ and a subgradient in parallel; sometimes analytically

result: can solve dual in time linear in number of (primal) variables

cf. standard cvx opt methods, cubic in number of primal variables

- with one coupling constraint, reduces to standard waterfilling method
- dual decomposition can be applied hierarchically

Integrality of bit allocation

first solve relaxed (convex) problem, then simple heuristic rounding

- relaxed problem gives bound on integer problem
- rounding or simple heuristic often close to optimal

simple choice: always round b_i down

- result is feasible (without network constraints),
- performance never more than factor of 4 suboptimal

Variable threshold rounding

- choose threshold parameter $t \in (0, 1]$
- round b_i as follows:

$$\tilde{b}_i = \begin{cases} \lfloor b_i \rfloor, & \text{if } b_i - \lfloor b_i \rfloor \leq t, \\ \lceil b_i \rceil, & \text{otherwise.} \end{cases}$$

- fix $\tilde{b}_i$, solve a convex feasibility problem over θ:

$$\begin{array}{rcl} f_k(\tilde{b}, \theta) & \leq & 0 \\ h_k^T \theta & \leq & d_k \\ \theta_k & \geq & 0 \end{array} \qquad (1)$$

- bisection on t to find largest $t^\star$ such that (1) is feasible

Example: A networked linear estimator

- $x \in \mathbf{R}^{20}$ to be estimated; we assume $\|x\| \leq 1$
- 200 sensors: $y_i = a_i^T x + v_i$, sensor noise v_i IID $\mathcal{N}(0, 10^{-6})$
- sensor signals transmitted over MAC; received signal: $y_{ri} = a_i^T x + v_i + q_i$
- scaling: $s_i = \|a_i\|$, uniformly distributed on $[0, 5]$

linear unbiased estimator: $\widehat{x} = K y_r$, where $KA = I$

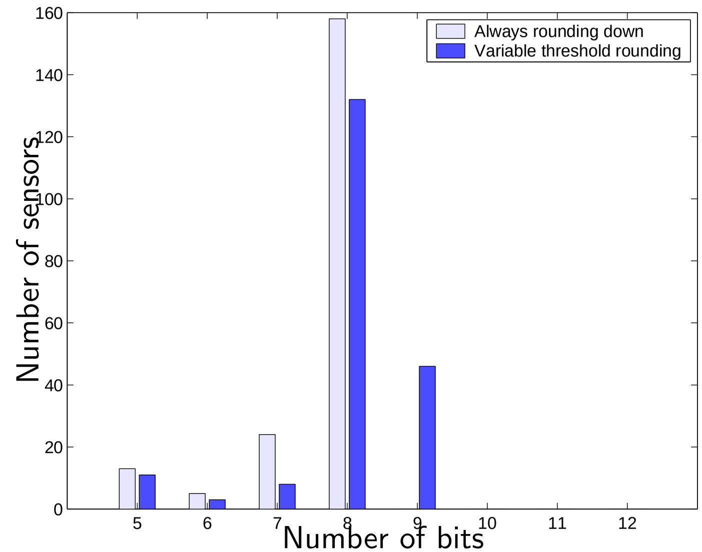
estimation error variance:

$$\mathbf{E}\,\|\widehat{x} - x\|^2 = \mathbf{Tr}\,K(R_q + R_v)K^T = \frac{1}{3}\sum_{i=1}^{M} s_i^2 \|k_i\|^2 2^{-2b_i} + \mathbf{Tr}(K R_v K^T)$$

we'll fix K as least-squares estimator $K = A^\dagger$
(minimizes estimation variance ignoring quantization noise)

Networked estimator (continued)

network constraints: total power limit, total bandwidth limit, $5 \leq b_i \leq 12$



Result: Signal-to-noise ratio same for all links of the MAC, hence bit allocation proportional to bandwidth and power allocation.

- **Uniform allocation:** $(b_i = 8)$, $\mathbf{rms}(x - \widehat{x}) = 3.6760 \times 10^{-3}$
- **Optimal (relaxed) allocation:** $\mathbf{rms}(x - \widehat{x}) = 3.1438 \times 10^{-3}$
- **Variable threshold rounding:** $\mathbf{rms}(x - \widehat{x}) = 3.2916 \times 10^{-3}$

The joint design problem

Allocate network resources *and* design linear system to minimize $\mathbf{E}\,\|z\|^2$

Issues:

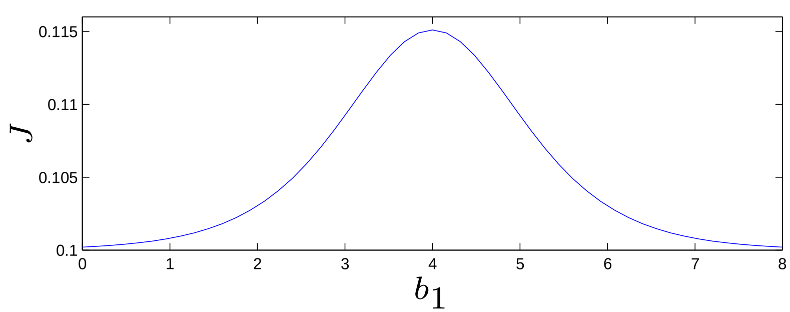
- optimal network resource allocation depends on choice of linear system
- choice of linear system affects scalings, hence quantization noise
- hence optimal choice of linear system depends on network resource allocation

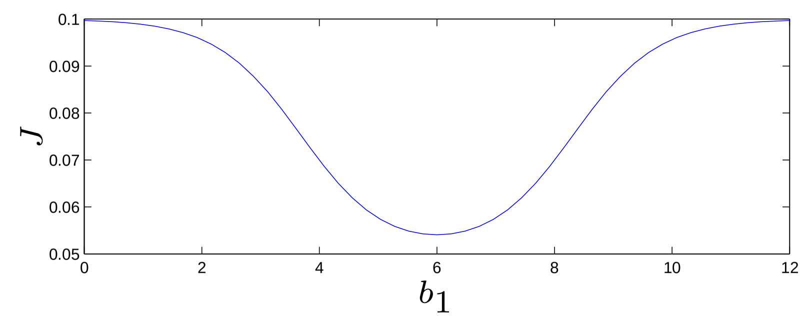
Joint design: simple example

estimate scalar x, with two sensors, total bit constraint $b_1 + b_2 = b_{\mathrm{tot}}$

$$y_1 = x + v_1, \qquad y_2 = x + v_2, \qquad \mathbf{E}\, v_1^2 = \mathbf{E}\, v_2^2 = 0.001$$

for fixed bit allocation, use optimal linear estimator; $J = \mathbf{E}(\widehat{x} - x)^2$





left: $b_1 + b_2 = 8$; assign all bits to one (either) sensor

right: $b_1 + b_2 = 12$: allocate bits equally

. . . complicated, highly non-convex problem

Iterative heuristic

heuristic algorithm for joint network/linear system design:

given initial system parameters, scalings, and network allocation

repeat

1. fix linear system parameters and scalings; find optimal network allocation
2. fix network parameters and scalings; find optimal linear system parameters
3. fix linear system and network parameters, find appropriate scalings

until convergence

Joint design of networked estimator

here, estimator K doesn't affect y, so scalings don't change

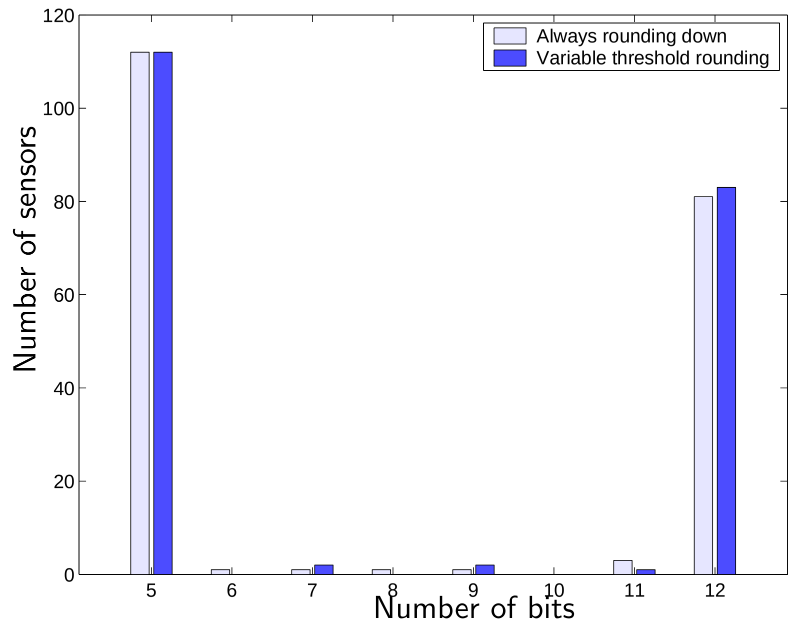
start with least-squares estimator, ignoring quantization noise

repeat

1. for fixed power/bandwidth allocation, take K as least-squares optimal (taking quantization noise into account)
2. fix estimator gain K and re-allocate power, bandwidth

until convergence

Distribution of bit allocation

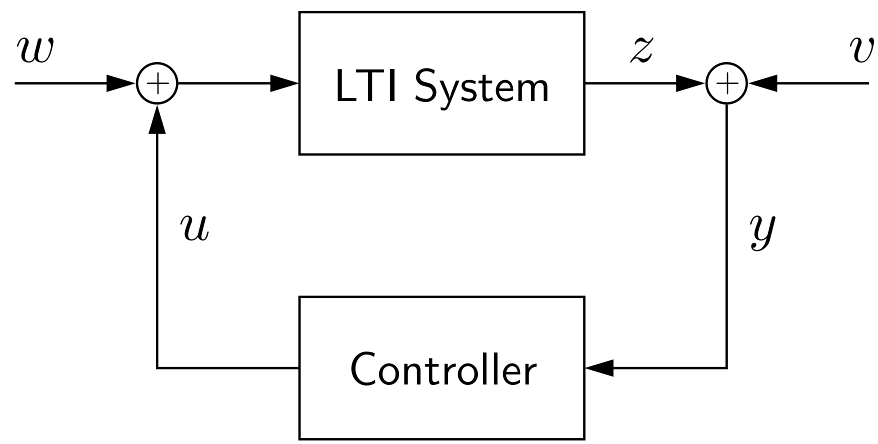


Again, SNR almost same for all links of the MAC, power and bandwidth allocation proportional to bit allocation.

Estimator performance

- **Uniform allocation:** $(b_i = 8)$, $\mathbf{rms}(x - \widehat{x}) = 3.6760 \times 10^{-3}$
- **Joint design, iterative heuristic:** $\mathbf{rms}(x - \widehat{x}) = 0.7210 \times 10^{-3}$
- **Variable threshold rounding:** $\mathbf{rms}(x - \widehat{x}) = 0.7221 \times 10^{-3}$

LQG control of dynamical system



linear system: $x(t+1) = Ax(t) + B\left(u(t) + w(t)\right)$, $y(t) = Cx(t) + v(t)$

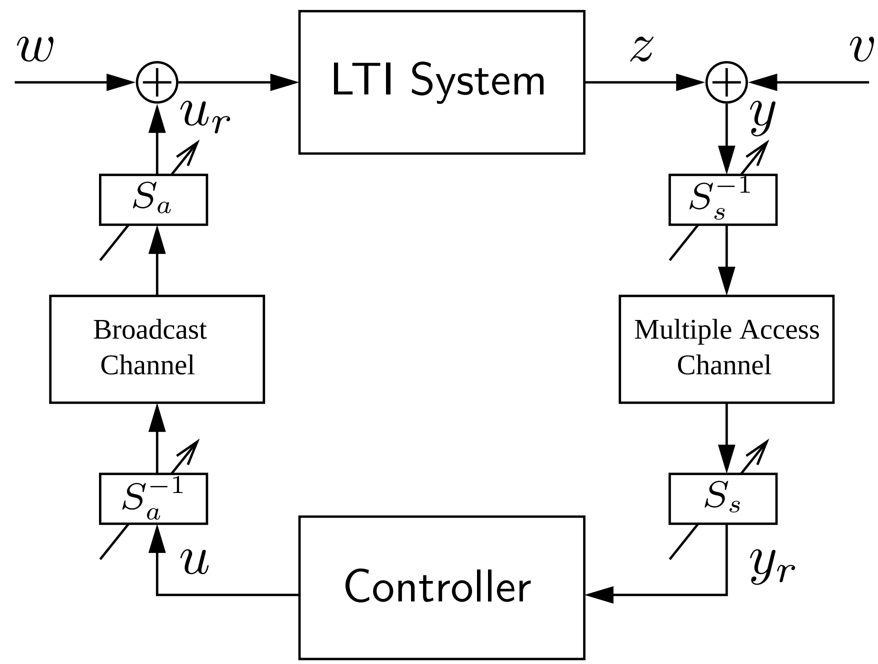
$w(t)$ and $v(t)$ IID white noises

actuator limits: $\mathbf{E}\, u_i^2 \leq 1$; performance signal: $z = Cx$

goal: minimize $\mathbf{E}\, \|z\|^2$ subject to actuator limits

optimal controller can be found using LQG theory, convex optimization

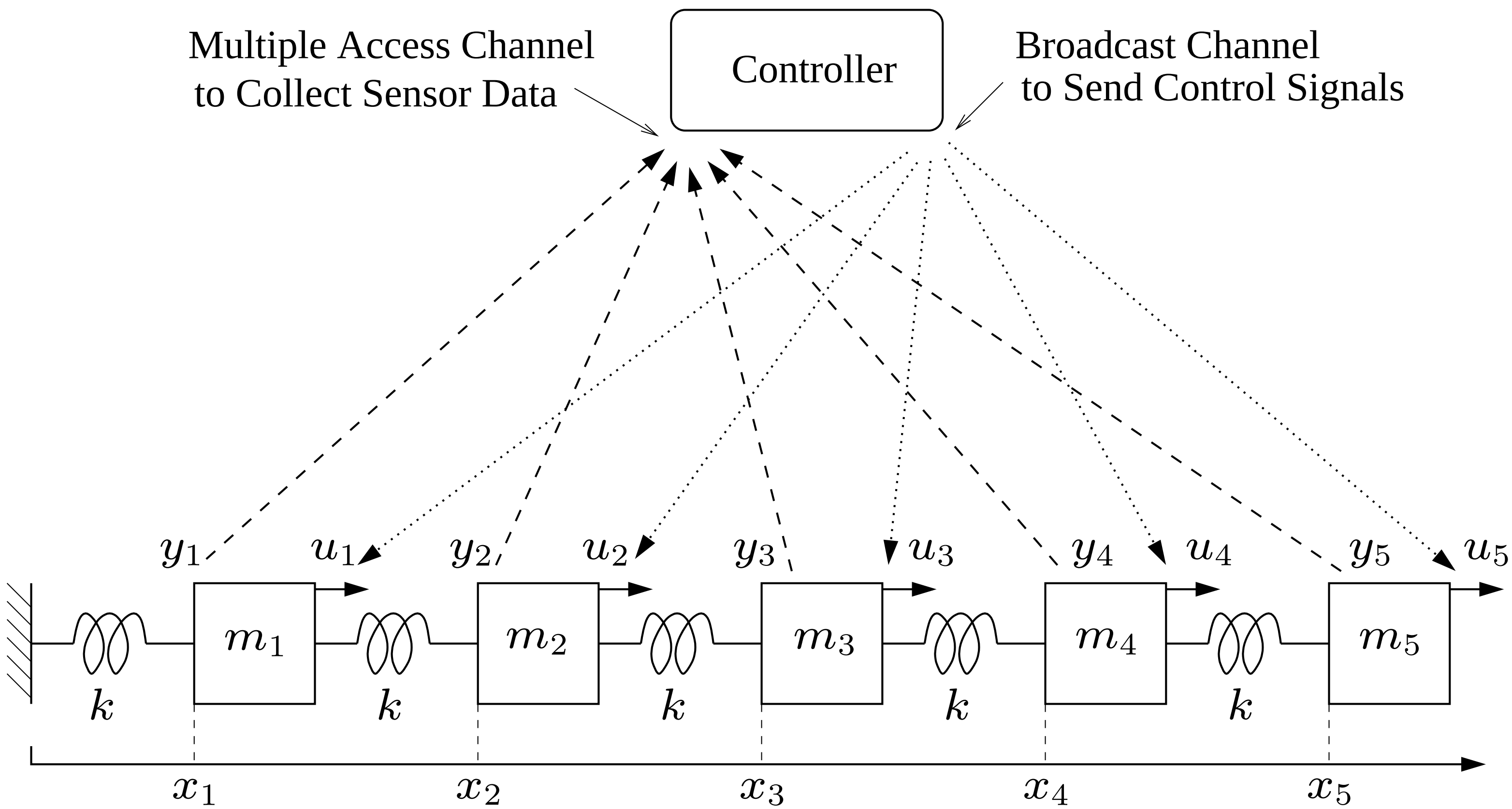
LQG control over communication networks



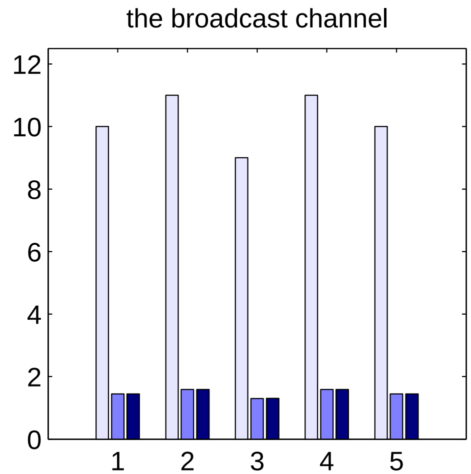
goal: allocate powers & bandwidth, design controller & scaling with

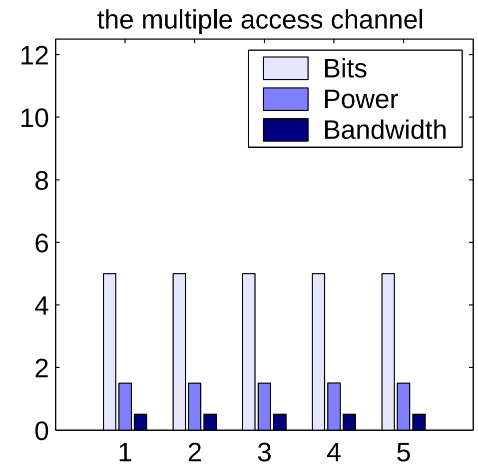
- **separate** sensor and actuator channel total power limits
- **combined** sensor and actuator channel bandwidth limit

Example: Mass-spring system



Solution via iterative heuristic:





Control system performance

- **uniform allocation**: P_i, W_i constant, $b_i = 8$ for all channels, results in: $\mathbf{rms}(u_i) = 1$ for all channels, $\mathbf{rms}(z) = 0.5493$
- **iterative heuristic**: $\mathbf{rms}(u_i) = 1$ for all channels, $\mathbf{rms}(z) = 0.1155$
- **iterative heuristic and rounding down**: see figure; $\mathbf{rms}(u_i) = 1$ for all channels, $\mathbf{rms}(z) = 0.1258$
- more bandwidth, hence more bits allocated to the actuator channel

Summary

for fixed linear system,

- optimal network resource allocation is convex problem (ignoring integrality)
- can efficiently solve via dual decomposition
- heuristic rounding works well

joint design problem

- complex, non-convex problem
- iterative heuristic works well
- outperforms optimization of network or system alone